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Heat Conduction Problem for an Finite Elliptical Cylinder

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Abstract- This paper contains a heat conduction problem for an finite elliptical cylinder to determine the temperature distribution with the help of Mathieu transform and Marchi-Fasulo transform techniques

Key Words- Heat conduction problem, Finite elliptic cylinder, Mathieu transforms, Marchi-Fasulo transform

Ams Subject Clasification No: 35-XXX. 44-XX. 80-XX.

1 INTRODUCTION

Integral transform technique plays important role in solving problem of applied Mathematics. Such problems have been out by **Sneddon** [3]. **Tranter** [5] and **Olcer** [2]. Hankel Transform is used to solve circular boundary value problems. Analogous to Hankel Transform, **Gupta** [1] and **Sharma** [4] have investigated finite Mathiue Transforms.

In this paper, we have generalized the problem of **Sneddon** [3]. We consider the heat conduction in a finite elliptic cylinder

2. STATEMENT OF THE PROBLEM

Heat conduction equation in elliptical co-ordinates (ξ,η) for elliptic cylinder as **Mclachlan** [8] is

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{2h^{-2}}{(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) + \frac{\partial^2 u}{\partial z^2} \tag{1}$$

$$u(\xi,\eta,z,o) = o, u(a,\eta,z,t) = f(\eta,z,t)$$

$$[u + k_1 \frac{\partial u}{\partial z}]_{z=-h} = 0$$

$$[u + k_2 \frac{\partial u}{\partial z}]_{z=h} = 0$$

Require result :

Finite Mathieu Transform

3 FINITE MATHIEU TRANSFORM

If the function T(ξ,η) is continuous and single valued in the region

$$0 \le \xi \le a, 0 \le \eta \le 2\pi. \, and \, \frac{\partial T}{\partial \xi} = 0 \; at \; \xi = a$$

then finite Mathieu transform is defined as

$$\bar{f}(q_{2n,m}) = \int_0^a \int_0^{2\pi} T(\xi,\eta)[\cosh 2\xi - \cos 2\eta].Ce_{2n}(\xi, q_{2n,m}) ce_{2n}(\eta, q_{2n,m}) d\xi d\eta \tag{2}$$

Where $q_{2n,m}$ is a root of the equation

$$Ce_{2n}(a,q) = 0$$

Then at any point within the range,

$$T(\xi,\eta) = \sum_{n=0}^{\infty} C_{2n} \; Ce_{2n}(\xi,q) \; ce_{2n}(\eta,q)$$

$$= \sum_{n=0}^{\infty}\left[\sum_{m=1}^{\infty} C_{2n} \; Ce_{2n}(\xi,q_{2n,m}).ce_{2n}(\eta,q_{2n,m})\right] \tag{3}$$

Where constant C_{2n} is

$$C_{2n} = \frac{\int_0^a \int_0^{2\pi} Ce_{2n}(\xi,q_{2n,m}) ce_{2n}(\eta,q_{2n,m}) u(\xi,\eta)[\cosh 2\xi - \cos 2\eta] d\xi d\eta}{\pi \int_0^a Ce_{2n}^2(\xi,q_{2n,m})[\cosh 2\xi - \theta_{2n,m}] d\xi} \tag{4}$$

$$= \frac{\bar{T}(q_{2n,m})}{\pi \int_0^a Ce_{2n}^2(\xi,q_{2n,m})[\cosh 2\xi - \theta_{2n,m}] d\xi} \tag{5}$$

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$$\theta_{2n,m} = \frac{1}{\pi}\int_0^{2\pi} Ce_{2n}^2(\eta, q_{2n,m})\cosh 2\eta \, d\eta \qquad (6)$$

INVERSION FORMULA OF MATHIUE TRANSFORM

The inversion formula for Mathiue Transform is given by

$$T(\xi,\eta) = \frac{\sum_{n=0}^{\infty}\left[\sum_{m=1}^{\infty}\bar{T}(q_{2n,m})(\xi, q_{2n,m}).ce_{2n}(\eta, q_{2n,m})\right]}{\pi\int_0^a Ce_{2n}^2(\xi, q_{2n,m})[\cosh 2\xi - \theta_{2n,m}]d\xi} \qquad (7)$$

PROPERTIES OF FINITE MATHIUE TRANSFORM

$$\int_0^a\int_0^{2\pi}\left(\frac{\partial^2 T}{\partial \xi^2}+\frac{\partial^2 T}{\partial \eta^2}\right)Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})d\xi \, d\eta \qquad (8)$$

Taking

$$I_1 = \int_0^a\int_0^{2\pi} Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})\frac{\partial^2 T}{\partial \xi^2}d\xi \, d\eta \qquad (9)$$

And

$$I_2 = \int_0^a\int_0^{2\pi} Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})\frac{\partial^2 T}{\partial \eta^2}d\xi \, d\eta$$

Now integration by parts we obtain

$$I_1 = \int_0^{2\pi} ce_{2n}(\eta, q_{2n,m})\left[Ce_{2n}(\xi, q_{2n,m})\frac{\partial T}{\partial \xi}Ce_{2n}(\xi, q_{2n,m})\right]_0^a d\eta$$

$$+\int_0^{2\pi}\int_0^a T\frac{\partial^2}{\partial \xi^2}\{Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})\}d\eta d\xi$$

$$I_1 = 2\pi\, A_0^{(2n)}\left[Ce_{2n}(\xi, q_{2n,m})\frac{\partial T}{\partial \xi} - T\frac{\partial}{\partial \xi}Ce_{2n}(\xi, q_{2n,m})\right]_0^a$$

$$+\int_0^{2\pi}\int_0^a T\frac{\partial^2}{\partial \xi^2}\{Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})\}d\eta d\xi$$

Since

$$Ce_{2n}(\xi, q_{2n,m}) = 0 \; and \; \frac{\partial T}{\partial \xi} = 0 \; at \; \xi = 0, \; \frac{\partial}{\partial \xi}Ce_{2n}(\xi, q_{2n,m}) = 0 \; at \; \xi = 0$$

$$I_1 = -2\pi\, A_0^{(2n)}TCe_{2n}(a, q_{2n,m}) + \int_0^{2\pi}\int_0^a T\frac{\partial^2}{\partial \xi^2}\{Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})\}d\eta d\xi$$

Similarly

$$I_2 = \int_0^a Ce_{2n}(\xi, q_{2n,m})\left[ce_{2n}(\eta, q_{2n,m})\frac{\partial T}{\partial \eta} - T\frac{\partial}{\partial \eta}ce_{2n}(\eta, q_{2n,m})\right]_0^{2\pi} d\xi$$

$$+\int_0^{2\pi}\int_0^a T\frac{\partial^2}{\partial \eta^2}\{Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})\}d\eta d\xi$$

Expression within limit vanishes at both limit.
Thus

$$\int_0^{2\pi}\int_0^a Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})\left(\frac{\partial^2 T}{\partial \xi^2}+\frac{\partial^2 T}{\partial \eta^2}\right)d\eta d\xi$$

$$I_1 = -2\pi\, A_0^{(2n)}uCe_{2n}(a, q_{2n,m}) + \int_0^{2\pi}\int_0^a T\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)V_{2n,m}(\xi,\eta)d\xi \, d\eta \qquad (10)$$

Where $V_{2n,m}(\xi,\eta)$ is solution of the equation

$$\frac{\partial^2}{d\xi^2}V_{2n,m} + \frac{\partial^2}{d\eta^2}V_{2n,m} + 2q_{2n,m}(\cosh 2\xi - \cos 2\eta)V_{2n,m} = 0$$

$V_{2n,m}$ *being equal to* $Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})$

Hence we get

$$\int_0^a\int_0^{2\pi}\left(\frac{\partial^2 T}{\partial \xi^2}+\frac{\partial^2 T}{\partial \eta^2}\right)Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})d\xi \, d\eta$$

$$= -2\pi\, A_0^{(2n)}TCe'_{2n}(a, q_{2n,m}) \quad (9) \qquad (11)$$

$$-2q_{2n,m}\int_0^{2\pi}\int_0^a T(Cosh 2\xi - \cos 2\eta)Ce_{2n}(\xi, q_{2n,m})ce_{2n}(\eta, q_{2n,m})d\xi \, d\eta$$

$$= -2\pi\, A_0^{(2n)}TCe'_{2n}(a, q_{2n,m}) - 2\bar{T}q_{2n,m} \qquad (12)$$

3. SOLUTION OF THE PROBLEM

Multiplying $(Cosh 2\xi - \cos 2\eta)Ce_{2n}(\xi, q_{2n,m}).Ce_{2n}(\eta, q_{2n,m})$ in equation (7) and integrating w.r.t. ξ from 0 to *a* and w.r.t η from 0 to 2π and using properties of Mathieu Transform (6) we get

$$\frac{d\bar{T}}{dt} = -2k\pi\, A_0^{(2n)}T(\xi,\eta,t)Ce'_{2n}(a, q_{2n,m}) - \frac{4q_{2n,m}}{h^2}.k.\bar{T} - a^2{}_n\bar{T}$$

$$or\; \frac{d\bar{T}}{dt} = -2k\pi\, A_0^{(2n)}T(\xi,\eta,t)Ce'_{2n}(a, q_{2n,m}) - k.\lambda_{2n,m}^2\bar{T} - a^2{}_n\bar{T} \qquad (13)$$

Where $\lambda_{2n,m}^2 = \frac{4q_{2n,m}}{h^2}$

$$or\; \frac{d\bar{T}}{dt} + k.\lambda_{2n,m}^2\bar{T} + a^2{}_n\bar{T} = -2k\pi\, A_0^{(2n)}T(\xi,\eta,t)Ce'_{2n}(a, q_{2n,m})$$

Which is an ordinary linear diff. Eqn.

$$I.F. = e^{\int(k.\lambda_{2n,m}^2 + a^2{}_n)dt} = e^{(k.\lambda_{2n,m}^2 + a^2{}_n).t}$$

And solution is

$$\bar{T}.e^{k\lambda_{2n,m}^2.t} = -2k\pi\, A_0^{(2n)}Ce'_{2n}(a, q_{2n,m})\int_0^t f(\eta,t).e^{(k\lambda_{2n,m}^2 + a^2{}_n).t}d\tau + A$$

$$at\; t = 0, \bar{T} = 0,\; A = 0$$

Hence $\bar{T} = -2k\pi\, A_0^{(2n)}Ce'_{2n}(a, q_{2n,m})\int_0^t f(\eta,t).e^{-(k\lambda_{2n,m}^2 + a^2{}_n).t}d\tau$

Using inversion theorem of Mathiue Transform, and

finite Marchi-Fasulo transform, we get

$$T(\xi,\eta,z,t)=\sum_{n=0}^{\infty}\frac{P_n(z)}{\lambda_n}\frac{-2k\pi\, A_0^{(2n)}Ce'_{2n}(a,q_{2n,m})\int_0^t f(\eta,t).e^{-k\lambda_{2n,m}^2.(t-\tau)}d\tau}{\pi\int_0^a \left[Ce_{2n}^2(\xi,q_{2n,m})\left(\cosh 2\xi-\theta_{2n,m}\right)\right]d\xi} \quad (14)$$

4. CONCLUSION

In this paper the temperature distributionthe of an elliptical cylinder have been determined with the help of finite Mathiue transform and Marchi-Fasulo transform techniques. The expressions are represented graphically. The results that are obtained can be applied to the structures or machines in engineering applications .

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temperature

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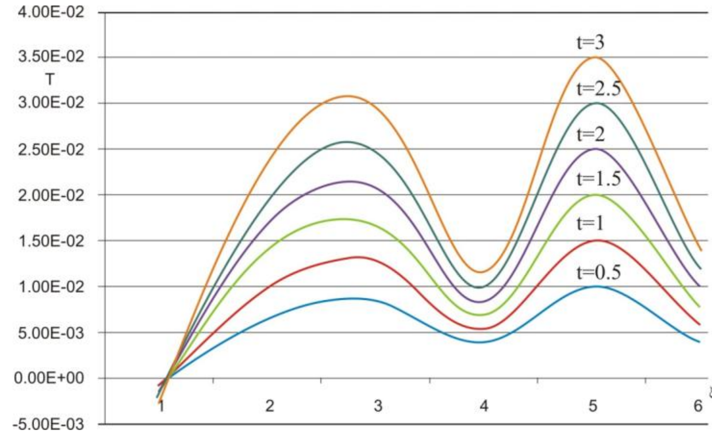
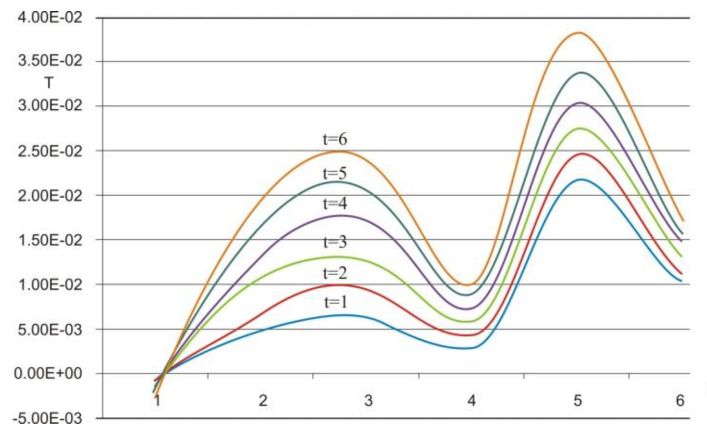


Graph T versus ξ for different value of t



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